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Sedimentation in a dispersion with vertical inhomogeneities

By FRANÇOIS FEUILLEBOIS
Laboratoire d'Aérothermique du Centre National de la Recherche Scientifique, F 92190 Meudon, France

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Statistical studies of hydrodynamic interactions between many particles in a dilute dispersion raise a problem of divergent integrals. This problem arises in particular when calculating the average velocity of sedimentation of solid spheres in a viscous fluid. The solution to this problem was given by Batchelor (1972) for monodisperse suspensions of spheres, on the basis of an assumption of homogeneity. This assumption is removed here. The problem of divergent integrals is reconsidered. The solution treats as successive steps:

(*a*) the average flow due to random statistically independent point forces;

(*b*) the average flow due to random statistically independent solid spheres, without hydrodynamic interactions;

(*c*) the average sedimentation velocity of random, pairwise-dependent solid spheres with hydrodynamic interactions, in a dilute suspension.

Considering the case of identical spheres, and assuming homogeneity in any horizontal plane, an expression is obtained for the average sedimentation velocity of a sphere in an otherwise inhomogeneous dispersion. The formula is written in terms of integrals involving probability distributions. It reduces, when the suspension is homogeneous, to a formula obtained by Batchelor.

The probability distributions are not calculated in this paper. In order to evaluate numerically the average velocity of sedimentation, a simple expression for the pair distribution function is assumed, and two different concentration profiles are considered, viz. a sinusoidal variation and a step function. In the case of sinusoidal concentration wave, it is found that the contribution of the inhomogeneity is, for small wavelengths, comparable in magnitude to that calculated for a homogeneous dispersion by Batchelor (1972), i.e. $-6.55c$.

The difference in velocity between the crest and trough of the wave is an increasing function of the wavelength. For a step function in concentration, particles at the top of the cloud start to fall faster, this effect being limited to a top layer about 10 radii thick.

For future study of the long-term behaviour of a sedimenting cloud, the evolution of the pair distribution function should be added to the present theory.

1. Introduction

The sedimentation of solid spherical particles in a viscous fluid has been studied in several recent papers. Herczynski & Pienkowska (1980) give a survey of different statistical approaches used to treat the hydrodynamics of a suspension of particles.

A basic method used in calculating hydrodynamic interactions between a large number of falling spheres was deduced by Batchelor (1972). It is based on the following assumptions:

the Reynolds number for the flow around the spheres is low;
the volume concentration of the spheres in the suspension is low;
the suspension is monodisperse;
it is homogeneous – the volume concentration of spheres is uniform.

The more general case of a polydisperse suspension was considered by Batchelor (1976) in a paper introducing the effects of Brownian motion. The detailed study of a polydisperse suspension was made recently by Batchelor (1982) and Batchelor & Wen (1982). Between these two studies, the case of spheres of equal radius, but different densities, without Brownian motion was treated by Feuillebois (1980), and the slightly different case of falling drops of different radii, but of the same density, was considered by Haber & Hetsroni (1981).

The assumption of a homogeneous suspension was essential in these papers, since it was the basis of some physical conditions introduced by Batchelor (1972) to calculate averages which would otherwise appear as divergent integrals. Other problems involving divergent integrals were studied along the same lines on the basis of the homogeneous-suspension assumption. Batchelor & Green (1972) calculated the viscosity of a suspension, Jeffrey (1973) derived the thermal conduction of a suspension, to quote only a few examples. The underlying method was presented by Jeffrey (1974) as a group expansion in terms of integrals involving successively larger numbers of particles. Later, using an averaged-equations approach, Hinch (1977) presented the method as a 'first renormalization' (as opposed to a 'second renormalization' generally applied to problems involving porous media). In each case, homogeneity is assumed.

In the present paper this assumption of a homogeneous suspension is removed. The problem of divergent integrals is reconsidered for inhomogeneous suspensions. For the case of batch sedimentation, an expression is obtained for the average velocity of a sphere, in terms of integrals involving probability distributions. This formula reduces to the one obtained by Batchelor (1972) when the suspension is homogeneous.

The ideas used here include the probability techniques of Batchelor and his coworkers and also some considerations about point singularities in Stokes flow (Saffman 1973; Chwang & Wu 1975; Felderhof 1976). A reason for introducing point singularities will be given now, together with the presentation of the calculation method and some definitions.

Consider a volume V containing a large number N of identical solid spheres. The volume V is large enough for the volume concentration c of the suspension to be low.

Using the notation of Batchelor (1972) for the statistics of suspensions, the probability density for one sphere to be centred at $\boldsymbol{x}$ (extremity of vector $\boldsymbol{x}$) is written as $P(\boldsymbol{x})$, and the probability density for N spheres to be centred at $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ respectively is written as $P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ or simply $P(\mathscr{C}_N)$. The symbol $\mathscr{C}_N$ stands for a configuration of the N spheres. The normalization conditions are

$$\int_V P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = N, \tag{1.1}$$

$$\int_{V^N} P(\mathscr{C}_N)\,\mathrm{d}\mathscr{C}_N = N!, \tag{1.2}$$

where $\mathrm{d}\boldsymbol{x}$ is the volume element in V and $\mathrm{d}\mathscr{C}_N = \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_N$ is the volume element in V^N. Given $N+1$ spheres, among which there is a 'test sphere' centred at $\boldsymbol{x}_0$, the conditional probability density to find the other spheres centred at $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ respectively is defined as

$$P(\mathscr{C}_N|\boldsymbol{x}_0) = P(\mathscr{C}_{N+1})/P(\boldsymbol{x}_0). \tag{1.3}$$

Our goal is the calculation of the average sedimentation velocity of the test particle at $\boldsymbol{x}_0$:

$$\bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0) = \frac{1}{N!}\int_{V^N} \boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \mathscr{C}_N)\, P(\mathscr{C}_N|\boldsymbol{x}_0)\, \mathrm{d}\mathscr{C}_N, \tag{1.4}$$

where $\boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \mathscr{C}_N)$ is the velocity of the test particle centred at $\boldsymbol{x}_0$, interacting with the other N spheres in the $\mathscr{C}_N$ configuration.

This familiar definition will be used for the present argument but a more general definition involving Schwartz's distributions (Schwartz 1966) will be used later in calculations.

The volume V is much larger than the volume of a sphere. Then, on the basis of the low concentration assumption, only pairs of spheres will be in the same vicinity. In view of the fact that all spheres are identical, (1.4) becomes to $O(c^2)$

$$\begin{aligned}\bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0) &= \sum_{k=1}^{N} \frac{1}{N}\int_V \boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \boldsymbol{x}_k)\, P(\boldsymbol{x}_k|\boldsymbol{x}_0)\, \mathrm{d}\boldsymbol{x}_k \\ &= \int_V \boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \boldsymbol{x}_1)\, P(\boldsymbol{x}_1|\boldsymbol{x}_0)\, \mathrm{d}\boldsymbol{x}_1,\end{aligned} \tag{1.5}$$

where $\boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ is the velocity of the sphere centred at $\boldsymbol{x}_0$ interacting with the sphere centred at $\boldsymbol{x}_1$. If V becomes infinitely large, it is known that $\boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ containing terms of order $1/r, 1/r^3$, with $r = |\boldsymbol{x}_1 - \boldsymbol{x}_0|$, decays slowly for $r \rightarrow \infty$. The conditional probability $P(\boldsymbol{x}_1|\boldsymbol{x}_0)$ tends to $P(\boldsymbol{x}_1)$ when the spheres centred at $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are far apart, and this last quantity is generally non-vanishing (for a homogeneous suspension it is a constant). Thus the integrand in (1.5) decays too slowly for the integral to be convergent. This familiar problem of 'divergent integral' (Batchelor 1972) will be reconsidered here.

First note that a similar problem arises in the calculation of the average fluid velocity at $\boldsymbol{x}_0$ due to spheres centred at $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, in particular for spheres that are statistically independent and non-interacting hydrodynamically, as we shall now see. This average fluid velocity is by definition

$$\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = \frac{1}{N!}\int_{V,\, \boldsymbol{x}_0 \in \mathrm{fluid}} \boldsymbol{v}(\boldsymbol{x}_0, \mathscr{C}_N)\, P(\mathscr{C}_N)\, \mathrm{d}\mathscr{C}_N, \tag{1.6}$$

where $\boldsymbol{x}_0 \in$ fluid means that $\boldsymbol{x}_0$ is in the fluid; $\boldsymbol{v}(\boldsymbol{x}_0, \mathscr{C}_N)$ is the fluid velocity at $\boldsymbol{x}_0$ due to all spheres in the $\mathscr{C}_N$ configuration. As there is neither statistical dependence nor interaction between the spheres, this average velocity can be written in terms of the direct effect of each of the N spheres behaving as if it were alone in the fluid. In view of the fact that all the spheres are identical, (1.6) becomes

$$\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = \sum_{k=1}^{N} \frac{1}{N}\int_{V,\, \boldsymbol{x}_0 \in \mathrm{fluid}} \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_k)\, P(\boldsymbol{x}_k)\, \mathrm{d}\boldsymbol{x}_k = \int_{V,\, \boldsymbol{x}_0 \in \mathrm{fluid}} \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1)\, P(\boldsymbol{x}_1)\, \mathrm{d}\boldsymbol{x}_1. \tag{1.7}$$

The fluid velocity $\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ at $\boldsymbol{x}_0$ due to a sphere centred at $\boldsymbol{x}_1$ is known to contain terms decreasing like $1/r$ and $1/r^3$ for $r = |\boldsymbol{x}_1 - \boldsymbol{x}_0| \rightarrow \infty$, and $P(\boldsymbol{x}_1)$ is non-vanishing, so that the integral (1.7) is divergent when V becomes infinitely large.

A 'divergent integral' also appears if we replace the independent solid spheres by independent point forces. For it is known that the fluid velocity at a point $\boldsymbol{x}$ located at a large distance from a sphere centred at $\boldsymbol{x}_1$ is identical, if terms of order $1/r$ are considered, with the fluid velocity due to a point-force singularity, or 'Stokeslet':

$$\left.\begin{aligned} \boldsymbol{v}_S &= \frac{\boldsymbol{\alpha}}{r'} + \frac{\boldsymbol{r}'(\boldsymbol{\alpha}\cdot\boldsymbol{r}')}{r'^3}, \\ p_S &= 2\mu\frac{\boldsymbol{\alpha}\cdot\boldsymbol{r}'}{r'^3}, \end{aligned}\right\} \tag{1.8}$$

where $\boldsymbol{v}_S$ is the fluid velocity, p_S the pressure, the subscript S stands for Stokeslet, μ is the fluid viscosity,

$$\boldsymbol{r}' = \boldsymbol{x} - \boldsymbol{x}_1, \quad r' = |\boldsymbol{r}'|, \tag{1.9}$$

and $\boldsymbol{\alpha}$ is the 'intensity' of the Stokeslet:

$$\boldsymbol{\alpha} = \frac{\boldsymbol{F}}{8\pi\mu}. \tag{1.10}$$

$\boldsymbol{F}$ is the force acting on the fluid at $\boldsymbol{x}_1$ (let the drag force acting on the sphere be $-\boldsymbol{F}$). The average fluid velocity is given by (1.7), where $\boldsymbol{v}$ is replaced by $\boldsymbol{v}_S$:

$$\bar{\boldsymbol{v}}(\boldsymbol{x}) = \int_V \boldsymbol{v}_S(\boldsymbol{x}, \boldsymbol{x}_1)\, P(\boldsymbol{x}_1)\, \mathrm{d}\boldsymbol{x}_1. \tag{1.11}$$

It is divergent when the volume V becomes infinitely large.

Thus 'divergent integrals' occur even when hydrodynamic interactions are absent. The present method for avoiding the 'divergent integrals' is based on two ideas.

(*a*) The treatment of divergent integrals is uncoupled from the calculation of hydrodynamic interactions. That this can be done will be proved in §5.

(*b*) The divergent integrals are avoided by keeping V finite. The justification for this is that, when hydrodynamic interactions are absent and spheres are statistically independent, the radius a of a sphere is not a relevant lengthscale (note that spheres may overlap). Thus when V/a^3 becomes very large, instead of keeping a finite, let us rather take a reference length based on V, and keep it finite. We then see the particles become very small in the limit $a^3/V \to 0$. In the first approximation, particles will behave like point forces (§§2, 3). In the next approximation, their small volume will be taken into account exactly (§4).

Our first problem is to calculate the average fluid velocity due to random point forces. We might initially think of using formula (1.11). However, this is not the formula to use, since we have to apply some boundary condition on the boundary (say ∂V) of V, and formula (1.11) does not necessarily allow that. The boundary condition that we will apply here is

$$\bar{\boldsymbol{v}} = 0 \quad \text{on } \partial V, \tag{1.12}$$

i.e. the no-slip condition on the walls of the container. Note that the lengthscale based on V is relevant since we have to apply the boundary condition on this scale. The process used to calculate the average velocity due to point forces in V will be first (in §2) to take a broad definition of the average, and derive equations for the average velocity. The mathematical treatment of point singularities in Stokes flow can be made rigorous in terms of the theory of distributions† (Schwartz 1966) and of the average defined by Gel'fand & Vilenkin (1964). The details will be published elsewhere

† No confusion should arise between probability distributions and Schwartz's distributions (or generalized functions).

(Feuillebois 1984) and results of this theory will be given here. We will then find a solution of these equations satisfying the required boundary condition (1.12). The integral (1.11) will actually be a particular solution of these equations, but not the one satisfying the boundary condition (1.12).

In §3 another problem involving point forces will be resolved, namely the sedimentation of a spherical cloud of point particles in an infinite fluid at rest. This particular configuration together with the one considered in §2 will be used in calculating the influence of the volume of the spheres (§4). The solid spheres considered are statistically independent, and are considered to fall in a viscous fluid without interactions between them.

In §5 the hydrodynamic interactions will be introduced together with the statistical dependence between pairs of spheres in a dilute suspension. An expression for the average velocity of sedimentation of a solid 'test' sphere will be obtained in terms of integrals involving probability distributions.

To simplify the analysis and the notation, the suspension is considered to be monodisperse, but the method can be extended without difficulty to a polydisperse suspension.

Finally in §6 we will assume some probability distributions so as to calculate numerically the integrals representing the inhomogeneous contributions to the average sedimentation velocity.

2. Average fluid velocity due to random independent point forces

The details of the arguments leading to the results presented in this section will be omitted. They were obtained on the basis of Schwartz's (1966) theory of distributions. The more accessible book by Schwartz (1979) was actually sufficient for our purpose. The work of Friedlander (1982) was also used. For more thorough description see Feuillebois (1984).

The equations that the average fluid velocity should satisfy are considered in three steps:

(*a*) the fluid velocity and pressure of a viscous fluid containing point-force singularities should satisfy the equations

$$\left.\begin{aligned} -\boldsymbol{\nabla} p+\mu\nabla^2\boldsymbol{v} &= -\sum_{k=1}^{N} \boldsymbol{F}(\boldsymbol{x}_K)\,\delta_{\boldsymbol{x}_K}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{v} &= 0. \end{aligned}\right\} \tag{2.1}$$

These are the Stokes equations with added point-force singularities. On the right-hand side of the momentum equation appear 'Dirac distributions' or 'δ-functions' at the different points $\boldsymbol{x}_K$ where the forces $\boldsymbol{F}(\boldsymbol{x}_K)$ are concentrated. Such equations can be found in the literature (e.g. Saffman 1973; Chwang & Wu 1975). Note that the stress tensor is discontinuous and therefore not derivable at the different points $\boldsymbol{x}_K$, in the sense of functions. But (2.1) can be shown to be true with derivatives defined in the sense of Schwartz's distributions.

(*b*) Now consider the points $\boldsymbol{x}_K$ to be random, so that $\boldsymbol{v}$ and $\boldsymbol{p}$ are also random. Our goal is to obtain equations for the averaged velocity and pressure. These equations will be obtained by averaging (2.1). But first the definition of the average should be refined as we are now dealing with distributions, and the average should commute with the derivatives in the sense of distributions. Also the average of the Dirac distribution has to be calculated.

A definition of the average of a distribution is given by Gel'fand & Vilenkin (1964). We adapt this definition to our problem. Consider a random distribution $T_{\mathscr{C}_N}$, i.e. a distribution that depends upon the random configuration $\mathscr{C}_N$ of the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$. Applying this distribution to a test function ϕ, we obtain the number

$$\langle T_{\mathscr{C}_N}(\boldsymbol{x}), \phi(\boldsymbol{x})\rangle. \tag{2.2}$$

This number depends on $\mathscr{C}_N$: it is a random variable, u say. This random variable can be averaged in the usual way, using a formula of the type

$$\bar{u} = \frac{1}{N!}\int_{V^N} u(\mathscr{C}_N)\, P(\mathscr{C}_N)\, \mathrm{d}\mathscr{C}_N, \tag{2.3}$$

or

$$\bar{u} = \frac{1}{N!}\int_{V^N} u(\mathscr{C}_N)\, P(\mathscr{C}_N|\boldsymbol{x}_0)\, \mathrm{d}\mathscr{C}_N, \tag{2.4}$$

depending upon the absence or presence of a test sphere centred at $\boldsymbol{x}_0$. The result obtained by this process provides a definition of the averaged distribution $\overline{T}$:

$$\langle \overline{T}, \phi\rangle = \bar{u} = \overline{\langle T, \phi\rangle}. \tag{2.5}$$

As compared with Gel'fand & Vilenkin's definition, it is assumed here that the probability distribution function may be written in term of a probability density P. We should emphasize here the physical requirement for such a function to exist: we exclude any event such that a particle sticks somewhere with a non-zero probability. We exclude thus any physical force allowing such an event to happen.

It can be checked that $\overline{T}$ defined by (2.5) is indeed a Schwartz distribution, and also that the average defined here commutes with the derivatives in the sense of distributions. For practical calculations, it is important to note that, when T is an integrable function, the average $\overline{T}$ defined by (2.5) is the same as the usual average of a function in terms of an integral (a formula of type (2.3) or (2.4)).

The average of the Dirac distribution is calculated using the definition (2.5) of the average and the classical definition of a Dirac distribution. In this section, we consider independent point forces and the definition (2.5) is to be completed with (2.3), where

$$P(\mathscr{C}_N) = \prod_{j=1}^{N} P(\boldsymbol{x}_j). \tag{2.6}$$

The result is

$$\bar{\delta}_{\boldsymbol{x}_j} = \frac{1}{N}\mathscr{P}(\boldsymbol{x}_j), \tag{2.7}$$

where

$$\mathscr{P}(\boldsymbol{x}_j) = \begin{cases} P(\boldsymbol{x}_j) & \text{for } \boldsymbol{x}_j \text{ in } V, \\ 0 & \text{otherwise.}\end{cases} \tag{2.8}$$

The averaged point-singularity 'Dirac distribution' looks like a continuous field. Formula (2.7) was given by Saffman (1973) for the case of a homogeneous suspension (his formula (4.1), where a missing Σ_α should be added).

(*c*) Using the definition (2.5), (2.3) of the average, and its mathematical properties, we can average the Stokes equations with random independent point-force singularities (2.1):

$$-\boldsymbol{\nabla}\bar{p} + \mu\nabla^2\bar{\boldsymbol{v}} = -\boldsymbol{F}\mathscr{P}(\boldsymbol{x}), \tag{2.9a}$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{v}} = 0. \tag{2.9b}$$

For identical particles in sedimentation, all the $\boldsymbol{F}(\boldsymbol{x}_K)$ are identical with a constant $\boldsymbol{F}$, and the bulk force on the right-hand side of (2.9*a*) resembles a gravity force in a fluid with variable density.

The average fluid velocity due to random independent point forces in a container of volume V is now calculated as the solution of (2.9) with the no-slip condition (1.12) on the boundary of V. The probability density is assumed as being a given function of position.

We remark that the averaged equations then have the trivial solution

$$\bar{v} = 0, \quad \nabla \bar{p} = \boldsymbol{F}\mathscr{P}(\boldsymbol{x}) \tag{2.10}$$

if $\boldsymbol{F}\mathscr{P}(\boldsymbol{x})$ is curl-free.

This is a hydrostatic field if the function $\mathscr{P}$ is such that the pressure gradient proportional to $\mathscr{P}(\boldsymbol{x})$ does not induce any fluid motion. This can be true for batch sedimentation in a vertical tank with $\mathscr{P}(\boldsymbol{x})$ varying only along the vertical coordinate, which is the case considered in this paper. Take the coordinate z pointing downwards. The stratified fluid of density $\mathscr{P}(z)$ is at rest if $\mathscr{P}$ is an increasing function of z.

The solution (2.10) may also be valid temporarily, with $\mathscr{P}$ being any function of z. We consider then a concentration profile which may eventually start to distort.

In the more general case of a fully inhomogeneous suspension, overturning may occur, as is the case in the problem of sedimentation in an inclined tube studied by Acrivos & Herbolzheimer (1979). Then the averaged equations (2.9) may not be sufficient to describe the general case: if the viscous term on the left-hand side of (2.9*a*) is large enough to balance the force term on the right-hand side, then the requirement that the Reynolds number on the lengthscale l of the volume V be small,

$$\frac{|\bar{v}|\, l}{\nu} \ll 1, \tag{2.11}$$

implies

$$\frac{|\boldsymbol{v}_{\mathrm{ps}}|\, a}{\nu} \ll \frac{1}{6\pi N}, \tag{2.12}$$

where

$$\boldsymbol{F} = 6\pi a \mu \boldsymbol{v}_{\mathrm{ps}}. \tag{2.13}$$

$\boldsymbol{v}_{\mathrm{ps}}$ is the limit sedimentation velocity of a single particle and $\mathscr{P} = O(N/V)$, $V = O(l^3)$.

The condition (2.12), for the Reynolds number relative to a particle, is very strong, since the number N of particles in V is very large. It will therefore seldom be met in practice. Thus, in the general case, for viscous terms to be important on the lengthscale l, the Reynolds number should be larger than unity on that scale, so that the inertia terms should be included in (2.9).

To conclude this section, as the homogeneities considered here are vertical, the average fluid velocity due to point force is identically zero, and so are its derivatives.

3. Sedimentation of a spherical cloud of point particles

Consider now the sedimentation of point particles located in a sphere of radius a and centre $\boldsymbol{x}_0$. Outside the sphere the fluid is clear, and the velocity of the fluid at infinity (at distances much larger than a) is zero. The fluid velocity and pressure should be continuous across the sphere surface. This problem of sedimentation of a finite cloud of point particles can be solved in two different ways, which we will now present.

The first way is purely mathematical. We use here results from §2. We want to solve the averaged equations (2.9) with

$$\mathscr{P}(\boldsymbol{x}) = \begin{cases} P(\boldsymbol{x}) & (|\boldsymbol{x}-\boldsymbol{x}_0| < a), \\ 0 & (|\boldsymbol{x}-\boldsymbol{x}_0| > a), \end{cases} \tag{3.1}$$

$$\bar{v} \to 0 \quad \text{as} \quad |\boldsymbol{x}-\boldsymbol{x}_0| \to \infty. \tag{3.2}$$

A solution is obtained by adding N Stokeslet fields like (1.8) and averaging using (2.3) and (2.5):

$$\bar{\boldsymbol{v}}(\boldsymbol{x}) = N\bar{\boldsymbol{v}}_S(\boldsymbol{x}) = \int_{r_1 < a} \left(\frac{\boldsymbol{\alpha}}{r'} + \frac{\boldsymbol{r}'(\boldsymbol{\alpha}\cdot\boldsymbol{r}')}{r'^3}\right) P(\boldsymbol{x}_1)\,\mathrm{d}\boldsymbol{x}_1, \tag{3.3}$$

where $\boldsymbol{r}' = \boldsymbol{x}-\boldsymbol{x}_1$, $r' = |\boldsymbol{r}'|$, $\boldsymbol{r}_1 = \boldsymbol{x}_1-\boldsymbol{x}_0$ and $r_1 = |\boldsymbol{r}_1|$.

This integral becomes vanishingly small when $\boldsymbol{x}$ is outside the sphere of centre $\boldsymbol{x}_0$ and radius a and a large distance from it. It is continuous as $\boldsymbol{x}$ crosses the sphere surface. The pressure p is calculated similarly.

More precisely, we will be interested (§4) in the average velocity and its Laplacian at the centre $\boldsymbol{x}_0$ of the sphere. For the homogeneous case

$$P(\boldsymbol{x}) = n, \tag{3.4}$$

where n is the number of point particles per unit volume. The integration of (3.3) is then straightforward in spherical polar coordinates, and we obtain at $\boldsymbol{x}_0$

$$\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = \tfrac{8}{3}\pi a^2 \boldsymbol{\alpha} n. \tag{3.5}$$

Now for the inhomogeneous case the average velocity at $\boldsymbol{x}_0$ can be rewritten as

$$\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = \tfrac{8}{3}\pi a^2 \boldsymbol{\alpha} P(\boldsymbol{x}_0) + \int_{r_1 < a} \left(\frac{\boldsymbol{\alpha}}{r_1} + \frac{\boldsymbol{r}_1(\boldsymbol{\alpha}\cdot\boldsymbol{r}_1)}{r_1^3}\right)[P(\boldsymbol{x}_1) - P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1. \tag{3.6}$$

Next, to calculate the Laplacian of the velocity, we use the fact that the average and the derivatives commute to write

$$\nabla^2 \bar{\boldsymbol{v}} = N\nabla^2 \bar{\boldsymbol{v}}_S = N\overline{\nabla^2 \boldsymbol{v}_S}. \tag{3.7}$$

Details of the following calculation can be found in Feuillebois (1984). The Laplacian of the Stokeslet velocity (1.8) in the sense of distributions yields†

$$\nabla^2 \boldsymbol{v}_S = -\tfrac{16}{3}\pi \boldsymbol{\alpha} \delta_{\boldsymbol{x}_1} + \mathrm{pv}\left(\frac{2\boldsymbol{\alpha}}{r'^3} - \frac{6\boldsymbol{r}'(\boldsymbol{\alpha}\cdot\boldsymbol{r}')}{r'^5}\right), \tag{3.8}$$

where the symbol pv denotes a principal-value distribution. The definition of such a distribution pv (f), where f is a function integrable except at the origin, is given in terms of its scalar product with a test function ϕ:

$$\langle \mathrm{pv}\,(f), \phi \rangle = \lim_{\epsilon \to 0} \int_{|\boldsymbol{x}| > \epsilon} f(\boldsymbol{x})\,\phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{3.9}$$

The average of (3.8) is calculated using the definition (2.5) of the average of a distribution and the averaged Dirac distribution (2.7), plus some standard theorems:

$$N\overline{\nabla^2 \boldsymbol{v}_S} = -\tfrac{16}{3}\pi \boldsymbol{\alpha} \mathscr{P}(\boldsymbol{x}) + \mathrm{pv} \int_V \left(\frac{2\boldsymbol{\alpha}}{r'^3} - \frac{6\boldsymbol{r}'(\boldsymbol{\alpha}\cdot\boldsymbol{r}')}{r'^5}\right) \mathscr{P}(\boldsymbol{x}_1)\,\mathrm{d}\boldsymbol{x}_1. \tag{3.10}$$

Evaluating this quantity at $\boldsymbol{x} = \boldsymbol{x}_0$, we use the fact that, for constant $\mathscr{P}$ (the homogeneous case), the principal-value distribution in (3.10) is then vanishing by symmetry. The result for the Laplacian of $\bar{\boldsymbol{v}}$ follows from (3.7):

$$\nabla^2 \bar{\boldsymbol{v}}(\boldsymbol{x}_0) = -\tfrac{16}{3}\pi \boldsymbol{\alpha} P(\boldsymbol{x}_0) + \mathrm{pv} \int_{r_1 < a} \left(\frac{2\boldsymbol{\alpha}}{r_1^3} - \frac{6\boldsymbol{r}_1(\boldsymbol{\alpha}\cdot\boldsymbol{r}_1)}{r_1^5}\right)[P(\boldsymbol{x}_1) - P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1. \tag{3.11}$$

If P is continuous at x_0 the principal-value symbol is no longer necessary since the

† I would like to thank Dr F. G. Friedlander for providing hints leading to this result.

integrand varies now as $1/r_1^2$ around $\boldsymbol{x}_1 = \boldsymbol{x}_0$ so that the integral is absolutely convergent.

In §4 we shall also need the quantity $\nabla^4 \bar{\boldsymbol{v}}(\boldsymbol{x}_0)$. This can be directly obtained from the averaged equations (2.9). With the definition (1.10) of $\boldsymbol{\alpha}$, we calculate

$$\nabla^2 \bar{p} = 8\pi\mu\boldsymbol{\alpha}\cdot\nabla\mathscr{P}, \tag{3.12}$$

$$\nabla^4 \bar{\boldsymbol{v}} = -8\pi\boldsymbol{\alpha}\nabla^2\mathscr{P} + 8\pi\boldsymbol{\alpha}\cdot\nabla\nabla\mathscr{P}. \tag{3.13}$$

This last quantity is zero, from the assumption that $\mathscr{P}$ varies along the vertical only.

In this problem of a spherical cloud of point particles, the velocity and its Laplacian and Laplacian square at the centre of the cloud have been obtained by direct calculations, using the theory of distributions. But these results may be obtained in another way for the particular case of a homogeneous dispersion. From (2.9) with conditions (3.1) and (3.2) we recognize the Hadamard problem of a drop of fluid falling in another fluid of different density.† The solution is classical (e.g. see Batchelor 1967, p. 235). It is important to note that the drop is found to be stable without any surface tension to maintain the spherical shape.

Let $\bar{\rho}$ be the density of the fluid inside the drop, and ρ the density of the fluid outside. The viscosities of both fluids are equal to μ. Let $\boldsymbol{g}$ be the acceleration due to gravity. In order to get an absolute pressure, we add to the flow field given by Batchelor (1967) a hydrostatic field inside the drop

$$\left.\begin{aligned} \boldsymbol{v}_{\mathrm{H}} &= 0, \\ \nabla p_{\mathrm{H}} &= (\bar{\rho}-\rho)\boldsymbol{g}. \end{aligned}\right\} \tag{3.14}$$

The resulting flow field inside the drop is

$$\bar{\boldsymbol{v}} = \boldsymbol{U} + \frac{1}{4}\left[\boldsymbol{U}\left(1-\frac{2r^2}{a^2}\right) + \frac{\boldsymbol{r}(\boldsymbol{U}\cdot\boldsymbol{r})}{a^2}\right], \tag{3.15}$$

$$\bar{p} = \tfrac{1}{3}(\bar{\rho}-\rho)\boldsymbol{g}\cdot\boldsymbol{r}, \tag{3.16}$$

where $\boldsymbol{U}$ is the sedimentation velocity of the drop:

$$\boldsymbol{U} = \frac{4}{15}\frac{a^2}{\mu}(\bar{\rho}-\rho)\boldsymbol{g}. \tag{3.17}$$

$\bar{\boldsymbol{v}}, \bar{p}$ satisfy (2.9) if the excess weight per unit volume of the interior fluid is identified with the force field due to the point particles:

$$(\bar{\rho}-\rho)\boldsymbol{g} = \boldsymbol{F}P(\boldsymbol{x}) = \boldsymbol{F}n. \tag{3.18}$$

From the preceding equations, and from (1.10), the velocity at the centre of the drop is found to be identical with the result (3.5). The Laplacian of the velocity at the centre of the drop can be calculated from the flow equations (2.9), using the expression (3.16) for the pressure. The result is identical with the part of (3.11) that corresponds to a homogeneous suspension.

4. Random independent spheres without hydrodynamic interactions

Until now we have only considered point forces, and have not taken the volumes of the particles into account. This volume effect will be considered here. The statistical dependence between the spheres and the hydrodynamic interactions will be considered

† I am grateful to Dr J. Rallison for this suggestion.

in §5. In other words, the spheres that are considered here can overlap, and they give the same flow field as if they were falling in isolation in the fluid.

The fluid velocity and pressure at a point $\boldsymbol{x}$ due to a sphere of radius a and centred at $\boldsymbol{x}_1$, falling with velocity $\boldsymbol{v}_{\mathrm{ps}}$, are

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{3a}{4}\left(\frac{\boldsymbol{v}_{\mathrm{ps}}}{r'} + \frac{\boldsymbol{r}'(\boldsymbol{v}_{\mathrm{ps}}\cdot\boldsymbol{r}')}{r'^3}\right) + \frac{a^3}{4}\left(\frac{\boldsymbol{v}_{\mathrm{ps}}}{r'^3} - \frac{3\boldsymbol{r}'(\boldsymbol{v}_{\mathrm{ps}}\cdot\boldsymbol{r}')}{r'^5}\right), \tag{4.1}$$

$$p(\boldsymbol{x}) = \tfrac{3}{2}\mu a \frac{\boldsymbol{v}_{\mathrm{ps}}\cdot\boldsymbol{r}'}{r'^3}, \tag{4.2}$$

where again $\boldsymbol{r}' = \boldsymbol{x}-\boldsymbol{x}_1$.

The force exerted on the fluid by the sphere is $6\pi a\mu\boldsymbol{v}_{\mathrm{ps}}$, and from (1.8)–(1.10) the sphere induces the same flow field at a point $\boldsymbol{x}$ as a Stokeslet of intensity

$$\boldsymbol{\alpha} = \tfrac{3}{4}a\boldsymbol{v}_{\mathrm{ps}}. \tag{4.3}$$

provided that $r' \gg a$.

The flow field at any point $\boldsymbol{x}$ outside the sphere can be also written in terms of the flow field due to a Stokeslet:

$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{S}} + \tfrac{1}{6}a^2\nabla^2\boldsymbol{v}_{\mathrm{S}}, \tag{4.4}$$

$$p = p_{\mathrm{S}} + \tfrac{1}{6}a^2\nabla^2 p_{\mathrm{S}}, \tag{4.5}$$

where $\boldsymbol{v}_{\mathrm{S}}$, p_{S} are given by (1.8). Thus, as was noticed by Hinch (1977), at any distance $r' > a$ the volume effect of the sphere can be simply taken into account by replacing the sphere by a Stokeslet plus a degenerate Stokeslet quadrupole at its centre.

For convenience we extend the expressions (4.4) and (4.5) for the fluid velocity and pressure to any point in space (any r'), the derivatives being then calculated in the sense of distributions.

Consider now N independent spheres. Each sphere induces the same flow field as if it were alone in the fluid, and can then be replaced without loss of generality by a Stokeslet plus a Stokeslet quadrupole located at its centre. The flow field $\boldsymbol{v}, p$, which we extend also inside the spheres, is then a solution of (after (2.1))

$$\left.\begin{aligned} -\nabla p + \mu\nabla^2\boldsymbol{v} &= -\boldsymbol{F}\sum_{k=1}^{N}(\delta_{\boldsymbol{x}_K} + \tfrac{1}{6}a^2\nabla^2\delta_{\boldsymbol{x}_K}),\\ \nabla\cdot\boldsymbol{v} &= 0. \end{aligned}\right\} \tag{4.6}$$

We can now reconsider the problem already mentioned in §1, i.e. to calculate the average fluid velocity at a point $\boldsymbol{x}_0$, whenever this point is in the fluid.

We are now in a better position to solve this problem. We first remark that $\boldsymbol{x}_0$ in the fluid means $|\boldsymbol{x}_0-\boldsymbol{x}_K| > a$ for any sphere centre $\boldsymbol{x}_K$, and that the spheres can be replaced without loss of generality by a Stokeslet plus a Stokeslet quadrupole at their centre. Then the idea is to calculate the average on configurations for which $|\boldsymbol{x}_0-\boldsymbol{x}_K| > a$ as the difference of two averages:

$$\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = \bar{\boldsymbol{v}}^{\mathrm{I}}(\boldsymbol{x}_0) - \bar{\boldsymbol{v}}^{\mathrm{II}}(\boldsymbol{x}_0), \tag{4.7}$$

where $\bar{\boldsymbol{v}}^{\mathrm{I}}(\boldsymbol{x}_0)$ is the average on configurations for which $\boldsymbol{x}_0$ is located anywhere with respect to the $\boldsymbol{x}_K$, and $\bar{\boldsymbol{v}}^{\mathrm{II}}(\boldsymbol{x}_0)$ is the average on configurations for which there is a K such that $|\boldsymbol{x}_0 - \boldsymbol{x}_K| < a$. We thus add, and then subtract, physical situations where

$\boldsymbol{x}_0$ is in a particle, and these situations are described, somewhat artificially, by considering $\boldsymbol{x}_0$ to be almost always in the fluid, i.e. either at the point singularities located at the centres of the spherical particles or in the fluid.

As we are dealing with distributions, we use instead of (1.6) the average in the sense of (2.3) and (2.5). $\bar{\boldsymbol{v}}^{\mathrm{I}}(\boldsymbol{x}_0)$ is defined with the probability density $P(\mathscr{C}_N)$, and $\bar{\boldsymbol{v}}^{\mathrm{II}}(\boldsymbol{x}_0)$ is defined with the probability density

$$P_{\mathrm{in}}(\mathscr{C}_N) = \begin{cases} P(\mathscr{C}_N) & (|\boldsymbol{x}_0-\boldsymbol{x}_K| < a; \quad K = 1,2,\ldots,N), \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Similar formulae can be written replacing the velocity by the pressure.

Each flow field $\bar{\boldsymbol{v}}^i, \bar{p}^i$ ($i = \mathrm{I}, \mathrm{II}$) is a solution of the equations obtained by averaging (4.6):

$$\left.\begin{aligned} -\nabla\bar{p}^i+\mu\nabla^2\bar{\boldsymbol{v}}^i &= -\boldsymbol{F}(\mathscr{P}^i+\tfrac{1}{6}a^2\nabla^2\mathscr{P}^i), \\ \nabla\cdot\bar{\boldsymbol{v}}^i &= 0, \end{aligned}\right\} \tag{4.9}$$

where

$$\mathscr{P}^i(\boldsymbol{x}) = \begin{cases} \left.\begin{aligned} &P(\mathscr{C}_N) \quad (i = \mathrm{I}) \\ &P_{\mathrm{in}}(\mathscr{C}_N) \quad (i = \mathrm{II}) \end{aligned}\right\} \text{ for } x \text{ in } V, \\ 0 \quad \text{otherwise.} \end{cases} \tag{4.10}$$

The boundary conditions are, for problem I (for which the relevant lengthscale is based on the volume V),

$$\bar{\boldsymbol{v}}^{\mathrm{I}} = 0 \quad \text{on } \partial V; \tag{4.11}$$

and for problem II (for which the relevant lengthscale is the radius a of the sphere that contains the point singularities)

$$\bar{\boldsymbol{v}}^{\mathrm{II}} \to 0 \quad \text{as} \quad |\boldsymbol{x}-\boldsymbol{x}_K| \to \infty. \tag{4.12}$$

Problems I and II have been solved in §§2 and 3 respectively, for the particular case of averaged point-force (Stokeslet) singularities. Here, we use the notation for these solutions $\bar{\boldsymbol{v}}^i_{\mathrm{F}}, \bar{p}^i_{\mathrm{F}}$ ($i = \mathrm{I}, \mathrm{II}$). The systems of equations already solved are

$$\left.\begin{aligned} -\nabla\bar{p}^i_{\mathrm{F}}+\mu\nabla^2\bar{\boldsymbol{v}}^i_{\mathrm{F}} &= -\boldsymbol{F}\mathscr{P}^i, \\ \nabla\cdot\bar{\boldsymbol{v}}^i_{\mathrm{F}} &= 0 \end{aligned}\right\} \quad (i = \mathrm{I}, \mathrm{II}). \tag{4.13}$$

As $\bar{\boldsymbol{v}}^{\mathrm{I}}_{\mathrm{F}}$ was found to be identically zero, the solution of problem I, (4.9)–(4.11), is obtained merely by applying the operator $1+\frac{1}{6}a^2\nabla^2$ to $\bar{\boldsymbol{v}}^{\mathrm{I}}_{\mathrm{F}}$:

$$\bar{\boldsymbol{v}}^{\mathrm{I}} = 0. \tag{4.14}$$

The solution of problem II, (4.9), (4.10) and (4.12), is obtained by applying the operator $1+\frac{1}{6}a^2\nabla^2$ to the solution from §3. The expression (4.7) for the average velocity then becomes

$$\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = -(\bar{\boldsymbol{v}}^{\mathrm{II}}_{\mathrm{F}}(\boldsymbol{x}_0)+\tfrac{1}{6}a^2\nabla^2\bar{\boldsymbol{v}}^{\mathrm{II}}_{\mathrm{F}}(\boldsymbol{x}_0)). \tag{4.15}$$

For the next section, the Laplacian $\nabla^2\bar{\boldsymbol{v}}(\boldsymbol{x}_0)$ will also be required. As (4.15) is actually valid for any $\boldsymbol{x}_0$, and since $\nabla^4\bar{\boldsymbol{v}}^{\mathrm{II}}_{\mathrm{F}}$ vanishes after (3.13), we obtain

$$\nabla^2\bar{\boldsymbol{v}}(\boldsymbol{x}_0) = -\nabla^2\bar{\boldsymbol{v}}^{\mathrm{II}}_{\mathrm{F}}(\boldsymbol{x}_0). \tag{4.16}$$

The average velocity $\bar{\boldsymbol{v}}^{\mathrm{II}}_{\mathrm{F}}$ and its Laplacian are given by (3.6) and (3.11) respectively.

The results (4.15) and (4.16) may then be written, using (4.3), as

$$\bar{v}(x_0) = -\tfrac{4}{3}\pi a^3 P(x_0)\, v_{ps} - \mathrm{pv}\int_{r_1<a} v(x_0, x_1)\,[P(x_1)-P(x_0)]\,\mathrm{d}x_1, \tag{4.17}$$

$$\tfrac{1}{6}a^2\nabla^2\bar{v}(x_0) = \tfrac{2}{3}\pi a^3 P(x_0)\, v_{ps} - \tfrac{1}{6}a^2\,\mathrm{pv}\int_{r_1<a} \{\nabla^2 v(x, x_1)\}_{x=x_0}\,[P(x_1)-P(x_0)]\,\mathrm{d}x_1. \tag{4.18}$$

In (4.17), $v(x_0, x_1)$ represents the velocity of the fluid at x_0 due to the sphere centred at x_1. It is given by (4.1) for $x = x_0$. By symmetry $x_0 - x_1$ can be replaced by

$$r_1 = x_1 - x_0 \tag{4.19}$$

in that formula.

In (4.18) the Laplacian appears in the usual sense of functions. We use Schwartz's notation (the braces) for derivatives in the sense of functions as opposed to derivatives in the sense of distributions.

This Laplacian becomes, from (4.1) and (4.19),

$$\{\nabla^2 v(x, x_1\}_{x=x_0} = \frac{3a}{2}\left(\frac{v_{ps}}{r_1^3} - \frac{3r_1(v_{ps}\cdot r_1)}{r_1^5}\right). \tag{4.20}$$

The first term in the result (4.17) may be obtained more directly for a homogeneous suspension. Considering the spheres as solid (i.e. not replaced by point singularities) we simply write that the velocity v is v_{ps} for a point in a sphere. As v_{ps} and P are constants, the volume of the sphere appears in the averaging integration. The equation

$$\bar{v} + \tfrac{4}{3}\pi a^3 P v_{ps} = 0 \tag{4.21}$$

was written by Batchelor (1972) as a condition that the fluid and particle mixture be at rest on average, or more precisely (Batchelor 1976) that it is at rest in a frame in which the average flux is zero, for a homogeneous suspension.

By the present approach, this condition is obtained as the solution of a system of equations, with the no-slip boundary condition on the boundary ∂V.

For the case of a homogeneous suspension, the equation (4.18) with only the first term on the right-hand side was obtained by Batchelor (1972) by noticing that the deviatoric stress tensor is constant on average and its divergence is thus zero on average. A term $\tfrac{1}{2}cv_{ps}$ was obtained in equation (3.13) of that paper (where $U_0 = v_{ps}$). It is identical with the first term on the right-hand side of (4.18) using the volume concentration

$$c = \tfrac{4}{3}\pi a^3 P. \tag{4.22}$$

The results (4.17) and (4.18) can be re-expressed for the case where the probability density, which varies along the vertical coordinate z only, can be developed as

$$P(x_1) = P(x_0) + z\frac{\mathrm{d}P}{\mathrm{d}z}(x_0) + \dots + \frac{z^m}{m!}\frac{\mathrm{d}^m P}{\mathrm{d}z^m}(x_0) + \dots, \tag{4.23}$$

where x_0 corresponds to $z = 0$. After some integral calculus we get

$$\bar{v}(x_0) = -\tfrac{4}{3}\pi a^3\left[P(x_0) + 3\sum_{k=1}^{\infty}\frac{a^{2k}}{(2k+1)!\,(2k+3)}\frac{\mathrm{d}^{2k}P}{\mathrm{d}z^{2k}}(x_0)\right]v_{ps}, \tag{4.24}$$

$$\tfrac{1}{6}a^2\nabla^2\bar{v}(x_0) = \tfrac{2}{3}\pi a^3\left[P(x_0) + 3\sum_{k=1}^{\infty}\frac{a^{2k}}{(2k+1)!\,(2k+3)}\frac{\mathrm{d}^{2k}P}{\mathrm{d}z^{2k}}(x_0)\right]v_{ps}, \tag{4.25}$$

$$\nabla^4\bar{v}(x_0) = 0. \tag{4.26}$$

5. Random pairwise-dependent interacting spheres

Consider $N+1$ spheres centred at $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ in the volume V. We are interested here in situations where one of the spheres is centred at a fixed point O, in order to average, for these situations, the velocity of this 'test sphere' over all possible locations of the other spheres.

A difficulty arises here: it is unlikely that there is any configuration for which the centre of one sphere is exactly at point O. A similar difficulty arises in statistical physics, where it is unlikely that a point in phase space, representative of the system of molecules, passes exactly through a point of the energy surface (Haar 1961). However, we can assume that there is a non-negligible number of configurations for which a centre $\boldsymbol{x}_0$ of a sphere is located in a small neighbourhood of point O. More precisely, we assume that the probability of the event $|\boldsymbol{x}_0| < \epsilon$ (where ϵ is a positive small number) is non-zero. Such an hypothesis may be called 'quasi-ergodic'. We define then the average velocity of the 'test sphere centred at point O' as

$$\bar{\boldsymbol{v}}_{\mathrm{p}}(O) \approx \bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0). \tag{5.1}$$

The average $\bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0)$ is defined by keeping the test sphere centred at $\boldsymbol{x}_0$, with the other spheres then taking all possible locations in V. As we are dealing with distributions, the definition (2.4), (2.5) of the average is used:

$$\langle \bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0), \phi\rangle = \frac{1}{N!}\int_{V^N} \langle \boldsymbol{v}_{\mathrm{p}}(\boldsymbol{x}_0, \mathscr{C}_N), \phi\rangle\, P(\mathscr{C}_N|\boldsymbol{x}_0)\,\mathrm{d}\mathscr{C}_N. \tag{5.2}$$

The definition (5.1), (5.2) differs in two ways from the more-straightforward definition (1.4) as given by Batchelor (1972). First for a homogeneous suspension, it was possible to take the centre $\boldsymbol{x}_0$ of a sphere to be fixed, as the choice of the point $\boldsymbol{x}_0$ is not relevant. But here, for an inhomogeneous suspension, the choice of the point O is relevant. In particular, we take the boundary ∂V into account, and this boundary would be random in a frame with $\boldsymbol{x}_0$ fixed, which would make integrations unpleasant. These difficulties are the reasons for the present quasi-ergodic hypothesis.

Secondly, we use a more general definition of the average in terms of distributions. This definition contains the ordinary definition (1.4) whenever $\boldsymbol{v}_{\mathrm{p}}$ is a function. But, as we introduced distributions as a intermediate mathematical tool, this refinement had to be introduced here.

This section differs from the preceding sections in that the hydrodynamic interactions between the spheres are introduced, together with the statistical dependence of the locations of the sphere centres. It will be useful to separate these different effects, rewriting (5.2) as

$$\begin{aligned}\langle \bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0), \phi\rangle = &\left\langle \frac{1}{N!}\int_{V^N} \boldsymbol{w}(\boldsymbol{x}_0, \mathscr{C}_N)\, P(\mathscr{C}_N|\boldsymbol{x}_0)\,\mathrm{d}\mathscr{C}_N, \phi\right\rangle \\ &+\left\langle \frac{1}{N!}\int_{V^N,\, \boldsymbol{x}_0\in \text{fluid}} [\boldsymbol{v}(\boldsymbol{x}_0, \mathscr{C}_N) + \tfrac{1}{6}a^2[\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \mathscr{C}_N)]_{\boldsymbol{x}=\boldsymbol{x}_0}]\right. \\ &\left.\times [P(\mathscr{C}_N|\boldsymbol{x}_0) - P(\mathscr{C}_N)]\,\mathrm{d}\mathscr{C}_N, \phi\right\rangle \\ &+\frac{1}{N!}\int_{V^N,\, \boldsymbol{x}_0\in \text{fluid}} \left\langle \boldsymbol{v}(\boldsymbol{x}_0, \mathscr{C}_N) + \tfrac{1}{6}a^2[\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \mathscr{C}_N)]_{\boldsymbol{x}=\boldsymbol{x}_0}, \phi\right\rangle P(\mathscr{C}_N)\,\mathrm{d}\mathscr{C}_N \\ &+\langle \boldsymbol{v}_{\mathrm{ps}}, \phi\rangle,\end{aligned} \tag{5.3}$$

where
$$\boldsymbol{w}(\boldsymbol{x}_0, \mathscr{C}_N) = \boldsymbol{v}_{\rm p}(\boldsymbol{x}_0, \mathscr{C}_N) - \boldsymbol{v}_{\rm ps} - \boldsymbol{v}(\boldsymbol{x}_0, \mathscr{C}_N) - \tfrac{1}{6}a^2[\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \mathscr{C}_N)]_{\boldsymbol{x}=\boldsymbol{x}_0}. \tag{5.4}$$

From Faxen's formula, the quantity

$$\boldsymbol{v}_{\rm ps} + \boldsymbol{v}(\boldsymbol{x}_0, \mathscr{C}_N) + \tfrac{1}{6}a^2[\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \mathscr{C}_N)]_{\boldsymbol{x}=\boldsymbol{x}_0}$$

equals the velocity of the sphere at $\boldsymbol{x}_0$ in the flow field created by the N spheres of the $\mathscr{C}_N$ configuration. The quantity $\boldsymbol{w}(\boldsymbol{x}_0, \mathscr{C}_N)$ then represents the effect of hydrodynamic interactions between the test sphere and the other spheres. The first integral in (5.3) represents the average effect of these interactions. The second integral in (5.3) contains the relative effect of the statistical dependence of the test sphere and the N other spheres, in the quantity

$$P(\mathscr{C}_N|\boldsymbol{x}_0) - P(\mathscr{C}_N).$$

For the hydrodynamic interactions, and for the statistical dependence between the sphere centres, the radius of a sphere is a relevant lengthscale. We then let the volume V become infinite, as an approximation to $V \gg a^3$. Since we will not use distributions but only functions, we have already written the first two terms in (5.3) as integrals on functions, applied to the test function ϕ. These integrals are convergent for pairwise-dependent spheres when V becomes infinite, as we will see shortly. Now the third term in (5.3) concerns independent spheres without hydrodynamic interactions, which is the problem solved in §4. As we used distributions for this solution, we keep the corresponding term in (5.3) in the general form of the average of a distribution.

To calculate the first and second integrals in (5.3), which extend now over the whole space $\mathbb{R}^{3N}$, we reduce them to integrals over the locations of a single sphere centre in $\mathbb{R}^3$, on the basis of the low-concentration assumption. This approach is the one of formula (2.10) of Batchelor (1972). Let us call $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ the resulting integrals:

$$\boldsymbol{J}_1 = \int_{\mathbb{R}^3} \boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{x}_1)\, P(\boldsymbol{x}_1|\boldsymbol{x}_0)\, \mathrm{d}\boldsymbol{x}_1 + O(c^2), \tag{5.5}$$

$$\boldsymbol{J}_2 = \int_{\mathbb{R}^3,\, x_0 \in \text{fluid}} [\boldsymbol{v}(\boldsymbol{x}_0\, \boldsymbol{x}_1) + \tfrac{1}{6}a^2[\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)]_{\boldsymbol{x}=\boldsymbol{x}_0}]\, [P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)]\, \mathrm{d}\boldsymbol{x}_1 + O(c^2), \tag{5.6}$$

with obvious notations for $\boldsymbol{w}$ and P in the case of one sphere centred at $\boldsymbol{x}_1$ together with the test sphere. Such an approximation is valid if the integrands decay fast enough for the integrals $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ to be convergent. The fact that the integrands decay fast enough results from the way (5.3) was constructed. But let us go into more detail.

In integral $\boldsymbol{J}_1$, the integrand is $O(1/r_1)^4$ for large r_1 because:

(*a*) it results from the behaviour of the mobility coefficients for two spheres (cf. Batchelor 1976) in that only the direct action of the sphere centred at $\boldsymbol{x}_1$ upon the sphere centred at $\boldsymbol{x}_0$ gives terms of larger order $(1/r_1, 1/r_1^3)$ in $\boldsymbol{v}_0(\boldsymbol{x}_0, \boldsymbol{x}_1)$;

(*b*) this direct action of the sphere centred at $\boldsymbol{x}_1$, i.e. the effect of the flow created by this sphere upon the velocity of the test sphere, can be expressed in term of Faxen's formula by

$$\boldsymbol{v}_{\rm ps} + \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2[\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)]_{\boldsymbol{x}=\boldsymbol{x}_0};$$

(*c*) $\boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ is defined by subtraction of these terms from $\boldsymbol{v}_{\rm p}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ and is thus $O(1/r_1)^4$;

(*d*) the probability density $P(\boldsymbol{x}_1|\boldsymbol{x}_0)$ appearing in (5.5) is bounded.

In integral $\boldsymbol{J}_2$, the velocity term decays like $1/r_1$. The behaviour of the probability term is obtained by using the normalization condition (1.1) for the probability density

$P(\boldsymbol{x}_1)$, together with the normalization condition for the conditional probability density:

$$\int_V P(\boldsymbol{x}_1|\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_1 = N.$$

Subtracting (1.1) from this equation, we obtain

$$\int_V [P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)]\,\mathrm{d}\boldsymbol{x}_1 = 0. \tag{5.7}$$

Equation (5.7) is valid for any V, in particular for infinite V. The integral on $\mathbb{R}^3$ is thus convergent. We expect that it is also absolutely convergent as there is no obvious physical reason why $P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)$ should change sign for increasingly larger r_1.

Nevertheless, a non-absolutely convergent integral is sufficient to prove, using Abel's lemma, that integral $\boldsymbol{J}_2$ (5.6) is absolutely convergent.

This demonstration is valid only if the probabilities can be expressed in terms of probability densities, as was assumed for the definition of the average (2.3)–(2.5). It was then observed that particles should not adhere anywhere with a non-zero probability. As an example of the opposite, let us consider electrically charged Brownian particles, which strongly repel each other. As the expansion of the cloud of N particles is limited by the walls of the container, it is of course possible that a number of particles may cling to these walls with a non-zero probability, and the present approach is not applicable in such circumstances. In particular, $P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)$ may then decrease too slowly for the integral (5.7) to be convergent.

The expression (5.3) of the average velocity of the test particle is now rewritten using (5.5), (5.6) and the results (4.17) and (4.18) for independent particles without interactions. We get, to order $O(c)$,

$$\begin{aligned}\bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0) = {} & \boldsymbol{v}_{\mathrm{ps}} - \tfrac{2}{3}\pi a^3 P(\boldsymbol{x}_0)\,\boldsymbol{v}_{\mathrm{ps}} \\ & - \mathrm{pv}\int_{r_1 < a} [\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}]\,[P(\boldsymbol{x}_1) - P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1 \\ & + \int_{r_1 > a} [\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}]\,[P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)]\,\mathrm{d}\boldsymbol{x}_1 \\ & + \int_{r_1 > 2a} \boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{x}_1)\,P(\boldsymbol{x}_1|\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_1. \end{aligned} \tag{5.8}$$

In the last integral we have specified that the spheres do not overlap, restricting the range of integration. The result (5.8) is analogous to formula (7.5) of Batchelor (1976). The new term appearing here, for homogeneous sedimentation, is the principal value of an integral over $r_1 < a$. Recall that the Laplacian in this integral is calculated in the sense of functions, and the principal-value symbol can be dropped whenever the probability is regular in $\boldsymbol{x}_0$. If we assume that the probability can be developed as in (4.23), then this integral can be rewritten, using (4.24) and (4.25), as

$$\begin{aligned} & -\int_{r_1 < a} [\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2 \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}]\,[P(\boldsymbol{x}_1) - P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1 \\ & \qquad = -2\pi a^3 \boldsymbol{v}_{\mathrm{ps}} \sum_{k=1}^{\infty} \frac{a^{2k}}{(2k+1)!\,(2k+3)} \frac{\mathrm{d}^{2k}P}{\mathrm{d}z^{2k}}(\boldsymbol{x}_0). \end{aligned} \tag{5.9}$$

This 'new term' involves all even derivatives of the probability $P(\boldsymbol{x})$ at the centre of the test sphere.

The effects of the inhomogeneity of the suspension on the average velocity of sedimentation come from the integral (5.9), but also indirectly through the variations of the probability densities which enter the integrals on $r_1 > a$ and $r_1 > 2a$ in (5.8).

Equation (5.8) can be simplified further, using the fact that the spheres do not overlap, to write the second integral as

$$\int_{r_1>a} [\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}][P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)]\,\mathrm{d}\boldsymbol{x}_1$$

$$= -\int_{a<r_1<2a} \boldsymbol{V}[P(\boldsymbol{x}_1) - P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1 - \mathrm{pv}\int_{r_1<2a} \boldsymbol{V}P(\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_1 + \mathrm{pv}\int_{r_1<a} \boldsymbol{V}P(\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_1 + \int_{r_1>2a} \boldsymbol{V}[P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)]\,\mathrm{d}\boldsymbol{x}_1, \quad (5.10)$$

where

$$\boldsymbol{V} = \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}. \quad (5.11)$$

The first integral on the right-hand side of (5.10) can be added to the first integral on the right-hand side of (5.8). The second and third integrals on the right-hand side of (5.10) can be calculated from the results of §4. Equation (5.8) then becomes

$$\bar{\boldsymbol{v}}_\mathrm{p}(\boldsymbol{x}_0) = \boldsymbol{v}_\mathrm{ps} - \tfrac{20}{3}\pi a^3 P(\boldsymbol{x}_0)\,\boldsymbol{v}_\mathrm{ps} - \mathrm{pv}\int_{r_1<2a} [\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}][P(\boldsymbol{x}_1) - P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1 + \int_{r_1>2a} [\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}][P(\boldsymbol{x}_1|\boldsymbol{x}_0) - P(\boldsymbol{x}_1)]\,\mathrm{d}\boldsymbol{x}_1 + \int_{r_1>2a} \boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{x}_1)\,P(\boldsymbol{x}_1|\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_1. \quad (5.12)$$

In this equation, $\boldsymbol{w}$ can be replaced by its definition

$$\boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{x}_1) = \boldsymbol{v}_\mathrm{p}(\boldsymbol{x}_0, \boldsymbol{x}_1) - \boldsymbol{v}_\mathrm{ps} - \boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) - \tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}. \quad (5.13)$$

For freely rotating spheres $\boldsymbol{v}_\mathrm{p}(\boldsymbol{x}_0, \boldsymbol{x}_1)$, the velocity of the particle centred at $\boldsymbol{x}_0$ interacting with the particle centred at $\boldsymbol{x}_1$, can be written in terms of the mobility coefficients A_{11}, A_{12}, B_{11} and B_{12} defined by Batchelor ('976):

$$\boldsymbol{v}_\mathrm{p}(\boldsymbol{x}_0, \boldsymbol{x}_1) = \left[(A_{11} + A_{12})\frac{\boldsymbol{r}_1\boldsymbol{r}_1}{r_1^2} + (B_{11} + B_{12})\left(\boldsymbol{I} - \frac{\boldsymbol{r}_1\boldsymbol{r}_1}{r_1^2}\right)\right]\cdot\boldsymbol{v}_\mathrm{ps}. \quad (5.14)$$

In Batchelor (1972), $A_{11} + A_{12}$ was denoted by λ_1 and $B_{11} + B_{12}$ by λ_2. After (4.1), (4.19) and (4.20):

$$\boldsymbol{v}(\boldsymbol{x}_0, \boldsymbol{x}_1) + \tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0} = \frac{3a}{4}\left(\frac{\boldsymbol{v}_\mathrm{ps}}{r_1} + \frac{\boldsymbol{r}_1(\boldsymbol{v}_\mathrm{ps}\cdot\boldsymbol{r}_1)}{r_1^3}\right) + \frac{a^3}{2}\left(\frac{\boldsymbol{v}_\mathrm{ps}}{r_1^3} - \frac{3\boldsymbol{r}_1(\boldsymbol{v}_\mathrm{ps}\cdot\boldsymbol{r}_1)}{r_1^5}\right). \quad (5.15)$$

Thus the expression (5.13) for $\boldsymbol{w}$ becomes

$$\boldsymbol{w}(\boldsymbol{x}_0, \boldsymbol{x}_1) = \boldsymbol{v}_\mathrm{ps}\frac{\mathscr{L}}{r_1} + \frac{\boldsymbol{r}_1(\boldsymbol{v}_\mathrm{ps}\cdot\boldsymbol{r}_1)}{r_1^2}\mathscr{M}, \quad (5.16)$$

where

$$\mathscr{L} = B_{11} + B_{12} - 1 - \frac{3a}{4r_1} - \frac{1}{2}\left(\frac{a}{r_1}\right)^3, \quad (5.17)$$

$$\mathscr{M} = A_{11} + A_{12} - B_{11} - B_{12} - \frac{3a}{4r_1} + \frac{3}{2}\left(\frac{a}{r_1}\right)^3. \quad (5.18)$$

Since A_{11}, A_{12}, B_{11} and B_{12} are functions of

$$s = r_1/a \tag{5.19}$$

only (Batchelor 1976), $\mathscr{L}$ and $\mathscr{M}$ are functions of s only.

Equation (5.12) is the main result of this paper. The average velocity of sedimentation of a particle centred at $\boldsymbol{x}_0$ is expressed in terms of the probability density at $\boldsymbol{x}_0$ (i.e. the number of particles per unit volume at $\boldsymbol{x}_0$) and of the conditional probability densities around $\boldsymbol{x}_0$. The term $-\frac{20}{3}\pi a^3 P(\boldsymbol{x}_0)\,\boldsymbol{v}_{\rm ps}$ and the first two integrals represent the contribution from the backflow induced by the sedimentation of the other particles: the downward flow of particles is balanced by a fluid flow upwards which has a retarding effect on the sedimentation of the test particle. Compared with the result for a homogeneous suspension, there is a new term due to the inhomogeneity of the backflow, which takes the form of the principal value of an integral. Note, however, that the inhomogeneity may also affect indirectly the other integrals through the probabilities. The last integral in (5.12) represents the influence of higher-order ($1/r^4$ and above) interactions between pairs of spheres.

In the result (5.12) everything is known except the probability densities. Obviously the ordinary probability density $P(\boldsymbol{x}_1)$ and conditional probability density $P(\boldsymbol{x}_1|\boldsymbol{x}_0)$ should satisfy some conservation equations and will probably be functions of the interactions between particles. But we leave this question for future study.

6. Examples of inhomogeneities. Numerical results

In this section we assume that the probability density distribution $P(\boldsymbol{x}_1)$ (or alternatively the concentration) varies along the vertical coordinate according to some specified function. We will choose first a sinusoidal variation, and then a step function as examples, and will calculate numerically the contributions of these inhomogeneities to the average velocity of sedimentation.

For the conditional probability density distribution we assume the following variation:

$$P(\boldsymbol{x}_1|\boldsymbol{x}_0) = \begin{cases} P(\boldsymbol{x}_1) & (r_1 > 2a), \\ 0 & (r_1 < 2a); \end{cases} \tag{6.1}$$

i.e. it is identical with the unconditional probability density distribution, except for the fact that two spheres do not overlap. For the case of a homogeneous dispersion (for constant $P(\boldsymbol{x}_1)$) Batchelor (1972) assumes the same variation (6.1) on the physical basis of the influence of a Brownian motion. Here this physical basis does not hold as we did not take the Brownian motion into account in the calculation of the average velocity of sedimentation, and this effect would be important for an inhomogeneous suspension. In general, the probabilities $P(\boldsymbol{x}_1|\boldsymbol{x}_0)$ and $P(\boldsymbol{x}_1)$ for $r_1 > 2a$ would probably differ owing to the interactions between the spheres and to Brownian motion in the inhomogeneous suspension. We leave these points for future study. Nevertheless we will assume (6.1) in order to get numerical estimates of the integrals appearing in (5.12).

For the case of a homogeneous dispersion, the second term on the right-hand side of (5.12) becomes

$$-5\boldsymbol{v}_{\rm ps}\,c,$$

where the volume concentration

$$c = \tfrac{4}{3}\pi a^3 P(\boldsymbol{x}_0). \tag{6.2}$$

The first two integrals on the right-hand side of (5.12) vanish, and the third integral was calculated by Batchelor (1972) as

$$-1.55\boldsymbol{v}_{\mathrm{ps}}\,c.$$

Thus, for an inhomogeneous suspension, using the assumption (6.1), (5.12) can be rewritten as

$$\begin{aligned}\bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0) &= \boldsymbol{v}_{\mathrm{ps}}(1-6.55c)\\ &\quad -\mathrm{pv}\int_{r_1<2a}[\boldsymbol{v}(\boldsymbol{x}_0,\boldsymbol{x}_1)+\tfrac{1}{6}a^2\{\nabla^2\boldsymbol{v}(\boldsymbol{x},\boldsymbol{x}_1)\}_{\boldsymbol{x}=\boldsymbol{x}_0}]\,[P(\boldsymbol{x}_1)-P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1\\ &\quad +\int_{r_1>2a}\boldsymbol{w}(\boldsymbol{x}_0,\boldsymbol{x}_1)\,[P(\boldsymbol{x}_1)-P(\boldsymbol{x}_0)]\,\mathrm{d}\boldsymbol{x}_1,\end{aligned}\qquad(6.3)$$

with c defined as in (6.2). Compared with Batchelor (1972), the present formula for the average velocity of sedimentation contains two integrals due to the inhomogeneity of the suspension, which we have now to calculate. Using the expression (5.16) for $\boldsymbol{w}$, (5.15), and the symmetry around the vertical axis, we integrate the triple integrals in (6.3) once:

$$\begin{aligned}\bar{\boldsymbol{v}}_{\mathrm{p}}(\boldsymbol{x}_0) &= \boldsymbol{v}_{\mathrm{ps}}(1-6.55c)-\mathrm{pv}\,2\pi a^2\boldsymbol{v}_{\mathrm{ps}}\int_{s=0}^{2}\int_{z_0-as}^{z_0+as}\left[\frac{3}{4s}+\frac{1}{2s^3}+\left(\frac{z_1-z_0}{as}\right)^2\left(\frac{3}{4s}-\frac{3}{2s^3}\right)\right]\\ &\quad\times[P(z_1)-P(z_0)]\,s\,\mathrm{d}z_1\,\mathrm{d}s\\ &\quad+2\pi a^2\boldsymbol{v}_{\mathrm{ps}}\int_{s=2}^{\infty}\int_{z_0-as}^{z_0+as}\left[\mathscr{L}(s)+\left(\frac{z_1-z_0}{as}\right)^2\mathscr{M}(s)\right][P(z_1)-P(z_0)]\,s\,\mathrm{d}z_1\,\mathrm{d}s,\end{aligned}\qquad(6.4)$$

where z_1, z_0 are the vertical components of $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ respectively, and s is defined as in (5.19). The functions $\mathscr{L}$ and $\mathscr{M}$ are given by (5.17) and (5.18). In these expressions appear the mobility coefficients for the problem of two spheres.

The problem of two spheres in sedimentation was solved by Adler (1981) and Jeffrey & Onishi (1984). The results of Jeffrey & Onishi provide the mobility coefficients in terms of series in $1/s$, which will be used in the present calculations. Dr D. Jeffrey kindly provided his numerical results in advance of their publication, and his computer programs. We used expansions of the mobility coefficients up to order s^{-220}. Let us quote only the first terms of these expansions:

$$\left.\begin{aligned}A_{11} &= 1-\frac{15}{4s^4}+\frac{11}{2s^6}+\frac{21}{2s^8}-\frac{167}{2s^{10}}+O(s^{-12}),\\ A_{12} &= \frac{3}{2s}-\frac{1}{s^3}+\frac{75}{4s^7}-\frac{15}{2s^9}+O(s^{-11}),\\ B_{11} &= 1-\frac{17}{16s^6}-\frac{5}{4s^8}-\frac{69}{16s^{10}}+O(s^{-12}),\\ B_{12} &= \frac{3}{4s}+\frac{1}{2s^3}+O(s^{-11}).\end{aligned}\right\}\qquad(6.5)$$

The advantage of using expansions in $1/s$ is that, for given P, the integrals in (6.4) can be integrated analytically, and the result is obtained as series, as we shall see. The REDUCE language for algebraic manipulations created by Hearn (1973) and its integration module designed by Norman (Cambridge University) was used as a help in the lengthy but straightforward integrations.

Two cases are considered for the probability distribution.

Case 1

The probability distribution is chosen as

$$P(z) = n^* + n^* A \cos \frac{2\pi z}{\lambda}, \tag{6.6}$$

where n^*, A and λ are constants. n^* is the average number density of particles in space, and the non-dimensional amplitude A is chosen to be $A \leqslant 1$ so that $P(z)$ is non-negative.

The result for the average velocity of sedimentation is

$$\bar{v}_{\mathrm{p}}(z_0) = v_{\mathrm{ps}}(1 + Sc - Gc^*), \tag{6.7}$$

where c, defined as in (6.2), is the local volume concentration at z_0,

$$c^* = \tfrac{4}{3}\pi a^3 n^* \tag{6.8}$$

is the average volume concentration in space, and

$$S = -6.55 + G_1 + G_2, \tag{6.9}$$

$$G = G_1 + G_2 \tag{6.10}$$

are sedimentation coefficients. The inhomogeneous coefficient G contains two terms. The coefficient G_1 comes from the first integral in (6.4). Note that the principal-value symbol may be dropped there, since $P(z)$ is continuous. The coefficient G_1 represents the inhomogeneous part of the contribution from the backflow. The coefficient G_2, which comes from the second integral in (6.4), represents the inhomogeneous part of the contribution from the higher-order interactions. Both G_1 and G_2 are functions of λ/a only.

The result for G_1 is

$$G_1 = 5 + \tfrac{15}{16}\Lambda^2 \cos 4\Lambda^{-1} - \tfrac{15}{64}\Lambda^3 \sin 4\Lambda^{-1}, \tag{6.11}$$

with

$$\Lambda = \frac{\lambda}{a\pi}. \tag{6.12}$$

Using the expansions of the mobility coefficients to order s^{-10}, as given in (6.5), the result for G_2 is

$$\begin{aligned} G_2 = {} & \tfrac{54253}{35840} - \tfrac{2044381}{1720320}\cos 4\Lambda^{-1} + \tfrac{83267}{4587520}\Lambda^3 \sin 4\Lambda^{-1} \\ & - \tfrac{83267}{1146880}\Lambda^2 \cos 4\Lambda^{-1} - \tfrac{408633}{2293760}\Lambda \sin 4\Lambda^{-1} \\ & - \tfrac{45}{8}\Lambda^{-1}\,\mathrm{si}\,4\Lambda^{-1} - \tfrac{321689}{860160}\Lambda^{-1}\sin 4\Lambda^{-1} - \tfrac{1262441}{1075200}\Lambda^{-2}\cos 4\Lambda^{-1} \\ & - \tfrac{53}{16}\Lambda^{-3}\,\mathrm{si}\,4\Lambda^{-1} + \tfrac{4452163}{3225600}\Lambda^{-3}\sin 4\Lambda^{-1} - \tfrac{75}{4}\Lambda^{-4}\,\mathrm{ci}\,4\Lambda^{-1} \\ & + \tfrac{175841}{806400}\Lambda^{-4}\cos 4\Lambda^{-1} + \tfrac{121}{120}\Lambda^{-5}\,\mathrm{si}\,4\Lambda^{-1} + \tfrac{38639}{403200}\Lambda^{-5}\sin 4\Lambda^{-1} \\ & - \tfrac{2}{9}\Lambda^{-6}\,\mathrm{ci}\,4\Lambda^{-1} + \tfrac{5413}{33600}\Lambda^{-6}\cos 4\Lambda^{-1} + \tfrac{5413}{8400}\Lambda^{-7}\,\mathrm{si}\,4\Lambda^{-1}. \end{aligned} \tag{6.13}$$

In (6.13) appear the special functions 'integral sine' si, and 'integral cosine' ci, defined as

$$\left.\begin{aligned} \mathrm{si}\,(x) &= -\int_x^{\infty} \frac{\sin t}{t}\,\mathrm{d}t, \\ \mathrm{ci}\,(x) &= -\int_x^{\infty} \frac{\cos t}{t}\,\mathrm{d}t. \end{aligned}\right\} \tag{6.14}$$

For large wavelengths, i.e. λ/a large, it can be checked that the large terms in (6.11) and (6.13) cancel out, by expanding the trigonometric and special functions (6.14) in Taylor series:

$$\left.\begin{aligned} G_1 &= 8\left(\frac{a\pi}{\lambda}\right)^2 + O\left(\frac{a}{\lambda}\right)^4, \\ G_2 &= A\,\frac{45}{16}\frac{a\pi^2}{\lambda} - \frac{138071}{9600}\left(\frac{a\pi}{\lambda}\right)^2 + \frac{53}{32}\left(\frac{a\pi}{\lambda}\right)^2\pi - \frac{75}{4}\left(\frac{a\pi}{\lambda}\right)^2 \ln\frac{4a\pi}{\lambda} + O\left(\frac{a}{\lambda}\right)^4, \end{aligned}\right\} \tag{6.15}$$

The higher-order terms have been omitted here for simplification. For infinite λ/a the suspension becomes homogeneous, and the inhomogeneous sedimentation coefficients G_1, G_2 vanish as required.

For small wavelengths, i.e. $\Lambda < 0.1$, more terms are required for the expansions of the mobility coefficients, in order to improve the precision on G_2. The expansions of the mobility coefficients up to $O(s^{-220})$ are used. In the term-by-term integration of the series appearing in (6.4) we use the following approximation formulae valid for large x:

$$\begin{aligned} s_{2n}(x) &= x^{2n}\int_x^\infty \frac{\sin x}{x^{2n}}\,\mathrm{d}x \\ &= \frac{(-1)^{n+1}}{(2n-1)!}\left\{\sum_{k=n-1}^{\infty}\frac{(-1)^k(2k+1)!}{x^{2(k-n)+2}}\cos x + \sum_{k=n}^{\infty}\frac{(-1)^{k+1}(2k)!}{x^{2(k-n)+1}}\sin x\right\}, \end{aligned} \tag{6.16a}$$

$$\begin{aligned} s_{2n+1}(x) &= x^{2n+1}\int_x^\infty \frac{\sin x}{x^{2n+1}}\,\mathrm{d}x \\ &= \frac{(-1)^{n+1}}{(2n)!}\left\{\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(2k)!}{x^{2(k-n)}}\cos x + \sum_{k=n}^{\infty}\frac{(-1)^{k+1}(2k+1)!}{x^{2(k-n)+1}}\sin x\right\}, \end{aligned} \tag{6.16b}$$

$$\begin{aligned} c_{2n}(x) &= x^{2n}\int_x^\infty \frac{\cos x}{x^{2n}}\,\mathrm{d}x \\ &= \frac{(-1)^{n+1}}{(2n-1)!}\left\{\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(2k)!}{x^{2(k-n)+1}}\cos x - \sum_{k=n-1}^{\infty}\frac{(-1)^k(2k+1)!}{x^{2(k-n)+2}}\sin x\right\}, \end{aligned} \tag{6.16c}$$

$$\begin{aligned} c_{2n+1}(x) &= x^{2n+1}\int_x^\infty \frac{\cos x}{x^{2n+1}}\,\mathrm{d}x \\ &= \frac{(-1)^{n+1}}{(2n)!}\left\{\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(2k+1)!}{x^{2(k-n)+1}}\cos x - \sum_{k=n}^{\infty}\frac{(-1)^{k+1}(2k)!}{x^{2(k-n)}}\sin x\right\}. \end{aligned} \tag{6.16d}$$

These formulae were derived using classical approximation formulae for si(x) ci(x), and integrating by parts. The expression of G_2 for small wavelengths is

$$G_2 = \sum_{j=3}^{219}\left\{u_j\, s_j\left(\frac{4}{\Lambda}\right) + v_j\, c_j\left(\frac{4}{\Lambda}\right) + w_j\right\}, \tag{6.17}$$

where

$$\left.\begin{aligned} u_j &= \frac{3}{8}\frac{\Lambda^3}{2^j}\left[\frac{2}{\Lambda}(l_{j+1} + m_{j+1}) - \Lambda m_{j-1}\right], \\ v_j &= \frac{3}{4}\frac{\Lambda^3}{2^j}m_j, \\ w_j &= -\frac{3l_{j+1} + m_{j+1}}{(j-2)\,2^{j-2}} \end{aligned}\right\} \tag{6.18}$$

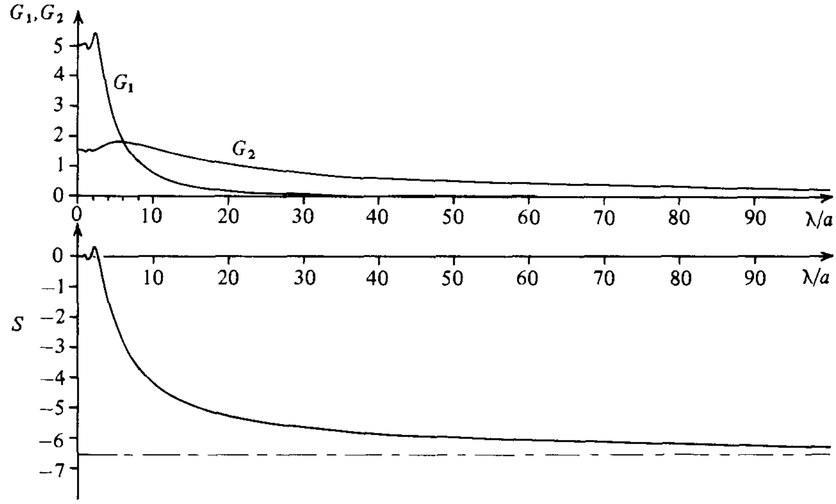

FIGURE 1. Case of a sinusoidal concentration wave (6.6). The inhomogeneous sedimentation coefficients G_1 and G_2, and the sedimentation coefficient S entering the expressions (6.7), (6.9) and (6.10) for the average velocity of sedimentation, versus the ratio of the concentration wavelength to the sphere radius.

and l_j and m_j are coefficients of s^{-j} in the expansions of $\mathscr{L}(s)$ and $\mathscr{M}(s)$. They are obtained from (5.17) and (5.18) and from the expansions of the mobility coefficients in $1/s$ provided by Jeffrey & Onishi (1984).

The inhomogeneous sedimentation coefficients G_1, G_2, and the sedimentation coefficient S (6.9) are plotted against λ/a in figure 1. It may be observed that the change in S due to the sinusoidal inhomogeneity can be important even for long wavelengths. For small wavelengths, G_1 and G_2 oscillate towards the limit values 5 and 1.55 respectively, S oscillates towards zero. The limit sedimentation velocity for very small wavelengths is

$$\bar{v}_p = v_{ps}(1-6.55c^*). \tag{6.19}$$

This behaviour is foreseeable, since a sinusoidal concentration profile of very small wavelength looks like a homogeneous field of concentration c^*, the average concentration in space.

Let us now consider the variation of $\bar{v}_p$ with z_0 (after (6.2), (6.6)–(6.8)). At $z_0 = \frac{1}{4}\lambda$, where $c = c^*$, the average velocity of sedimentation is expressed by (6.19). The inhomogeneity of the suspension exercises no influence at this point. More generally, for a probability distribution $P(z)$ that is antisymmetric about the plane $z = z_0$ it can be remarked that the inhomogeneous contribution to the average velocity of sedimentation cancels out. This is because the fluid velocity at $\boldsymbol{x}_1$ due to a sphere centred at $\boldsymbol{x}_0$ is symmetric about the plane $z = z_0$, and the mobility coefficients for two spheres centred at $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ respectively are symmetric when $\boldsymbol{x}_1 - \boldsymbol{x}_0$ is changed to $-(\boldsymbol{x}_1 - \boldsymbol{x}_0)$. Then the integrals in (6.4) cancel out.

The influence of the inhomogeneous sedimentation coefficients G_1 and G_2 reaches a maximum at the crest and trough of the concentration wave. The difference between the average velocities of sedimentation at these two points, which is the rate of overturning, is

$$\bar{v}_p(0) - \bar{v}_p(\tfrac{1}{2}\lambda) = 2Ac^* v_{ps} S. \tag{6.20}$$

For vanishingly small wavelengths, S vanishes, and overturning does not take place. For very large wavelengths, S decays to -6.55, and the rate of overturning is at a maximum.

Note that if the expression of the average velocity of sedimentation in a homogeneous dispersion (Batchelor 1972) were used to calculate the rate of overturning the result would be (6.20) with $S = -6.55$. This simplification would not then take into account the fact that the rate of overturning varies according to the wavelength.

Case 2

The probability distribution is chosen as

$$P(z) = \begin{cases} 0 & (z < 0), \\ n & (z > 0), \end{cases} \tag{6.21}$$

where n is a constant number of particles per unit volume, and the z-axis is pointing downwards.

In the calculation of the first integral in (6.4), the principal-value symbol can be dropped, except for the case $z_0 = 0$. Let us now consider this case. Using the polar angle θ between $\boldsymbol{x}_1 - \boldsymbol{x}_0$ and the vertical pointing downwards, the integral can be rewritten, in the case $z_0 \to 0+$,

$$-2\pi a^3 \boldsymbol{v}_{\mathrm{ps}}\, \mathrm{pv} \int_{s=0}^{2} \int_{\theta=0}^{\pi} \left[\frac{3}{4s} + \frac{1}{2s^3} + \left(\frac{3}{4s} - \frac{3}{2s^3}\right)\cos^2\theta\right] \left[P(as\cos\theta) - n\right] s^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}s.$$

From the definition (3.9) of a principal value, integration on θ is performed first. (In fact, for a fixed $\theta > \frac{1}{2}\pi$, the integral on s would be divergent).

The result for the average velocity of sedimentation can be written in the form

$$\bar{\boldsymbol{v}}_{\mathrm{p}}(z_0) = \boldsymbol{v}_{\mathrm{ps}}(1 + Sc), \tag{6.22}$$

where c is again defined as in (6.2) and S as in (6.9).

G_1 and G_2, coming respectively from the first and the second integral in (6.4), are now

(i) for $z_0/a \leqslant 2$

$$G_1 = \frac{5}{2} - \frac{15}{8}\frac{z_0}{a} + \frac{5}{32}\left(\frac{z_0}{a}\right)^3, \quad G_2 = G_2^{(0)} + G_2^{(1)}\frac{z_0}{a} + G_2^{(3)}\left(\frac{z_0}{a}\right)^3; \tag{6.23}$$

(ii) for $z_0/a > 2$

$$\left.\begin{aligned} G_1 &= 0, \\ G_2 &= \frac{45}{32}\frac{a}{z_0} - \frac{53}{128}\left(\frac{a}{z_0}\right)^3 - \frac{225}{224}\left(\frac{a}{z_0}\right)^4 - \frac{121}{320}\left(\frac{a}{z_0}\right)^5 + \frac{5}{24}\left(\frac{a}{z_0}\right)^6 + \sum_{m=0}^{M-10} G_{2(m)}\left(\frac{a}{z_0}\right)^{m+7}. \end{aligned}\right\} \tag{6.24}$$

The coefficients $G_2^{(0)}$, $G_2^{(1)}$ and $G_2^{(3)}$ are found as series:

$$\left.\begin{aligned} G_2^{(0)} &= \tfrac{903}{1280} - \sum_{m=0}^{M-10} 2^{-m}[a_{11}(m+10) + a_{12}(m+10) + 2b_{11}(m+10) + 2b_{12}(m+10)] \\ &\quad \times [\tfrac{1}{256}m^5 + \tfrac{43}{256}m^4 + \tfrac{23}{8}m^3 + \tfrac{1567}{64}m^2 + \tfrac{415}{4}m + 175]/D, \\ G_2^{(1)} &= -\tfrac{61}{2048} + \sum_{m=0}^{M-10} 2^{-m}[b_{11}(m+10) + b_{12}(m+10)] \\ &\quad \times [\tfrac{3}{512}m^5 + \tfrac{63}{256}m^4 + \tfrac{2103}{512}m^3 + \tfrac{4359}{128}m^2 + \tfrac{17955}{128}m + \tfrac{3675}{16}]/D, \\ G_2^{(3)} &= -\tfrac{2833}{344064} + \sum_{m=0}^{M-10} 2^{-m}[a_{11}(m+10) + a_{12}(m+10) - b_{11}(m+10) - b_{12}(m+10)] \\ &\quad \times [\tfrac{1}{2048}m^5 + \tfrac{5}{256}m^4 + \tfrac{637}{2048}m^3 + \tfrac{2525}{1024}m^2 + \tfrac{623}{64}m + \tfrac{245}{16}]/D, \end{aligned}\right\} \tag{6.25}$$

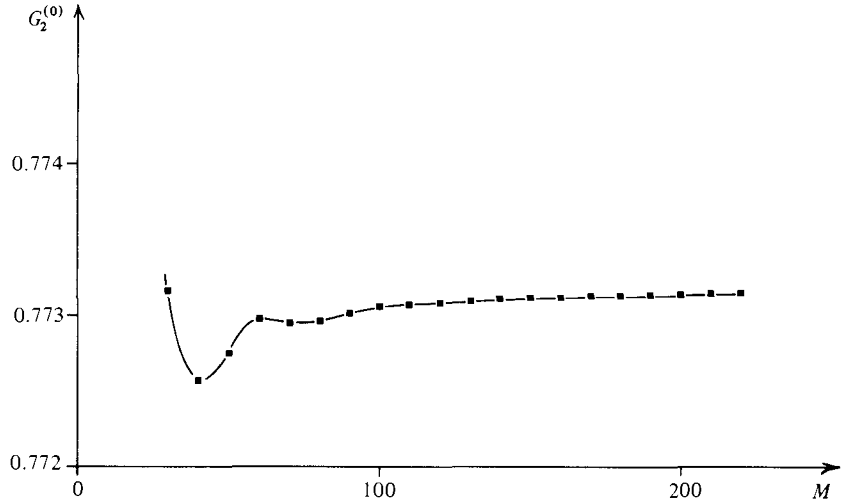

FIGURE 2. The result for $G_2^{(0)}$ as a function of the number M of terms used in the series (6.27) for the mobility coefficients.

with

$$D = m^6+50m^5+1037m^4+11420m^3+70436m^2+230420m+313600. \quad (6.26)$$

In (6.25) the coefficients a_{11}, a_{12}, b_{11} and b_{12} are those appearing in the series

$$\left.\begin{aligned} A_{11} &= 1+\sum_{m=1}^{M} a_{11}(m)\,s^{-m}, \quad A_{12} = \sum_{m=1}^{M} a_{12}(m)\,s^{-m}, \\ B_{11} &= 1+\sum_{m=1}^{M} b_{11}(m)\,s^{-m}, \quad B_{12} = \sum_{m=1}^{M} b_{12}(m)\,s^{-m}. \end{aligned}\right\} \quad (6.27)$$

These series up to $m = 9$ were used to evaluate the first fractional terms in (6.25). The calculation of the sums in (6.25) required the knowledge of the coefficients $a_{11}(m+10)$, $a_{12}(m+10)$, ... (for $m = 0, 1, ..., M-10$) provided by Jeffrey & Onishi (1984). We used the expansions (6.27) up to $M = 220$, and obtained

$$G_2^{(0)} = 0.773, \quad G_2^{(1)} = -0.0341, \quad G_2^{(3)} = -0.01322. \quad (6.28)$$

The precision on these coefficients was estimated by plotting the series against different values of M, up to $M = 220$, as shown on figure 2 for $G_2^{(0)}$.

The general coefficient in the series (6.24) was found to be

$$\begin{aligned} G_{2(m)} = -\{&[a_{11}(m+10)+a_{12}(m+10)]\,[\tfrac{3}{2}m^4+\tfrac{99}{2}m^3+609m^2+3312m+6720] \\ &+[b_{11}(m+10)+b_{12}(m+10)]\,[3m^3+75m^2+618m+1680]\}/D, \end{aligned} \quad (6.29)$$

with D as in (6.26).

The inhomogeneous sedimentation coefficients G_1 and G_2 ((6.23) and (6.24)) and the sedimentation coefficient S calculated from these by (6.9) are plotted on figure 3. It can be checked, by using the integrals in (6.4) and performing some analytical calculations, that G_1 and G_2 are continuous at $z_0/a = 2$ and have continuous first derivatives there.

A test sphere centred at $z_0 = 0$ is interacting with a semi-infinite homogeneous suspension. As the influences of other spheres are additive, the sedimentation

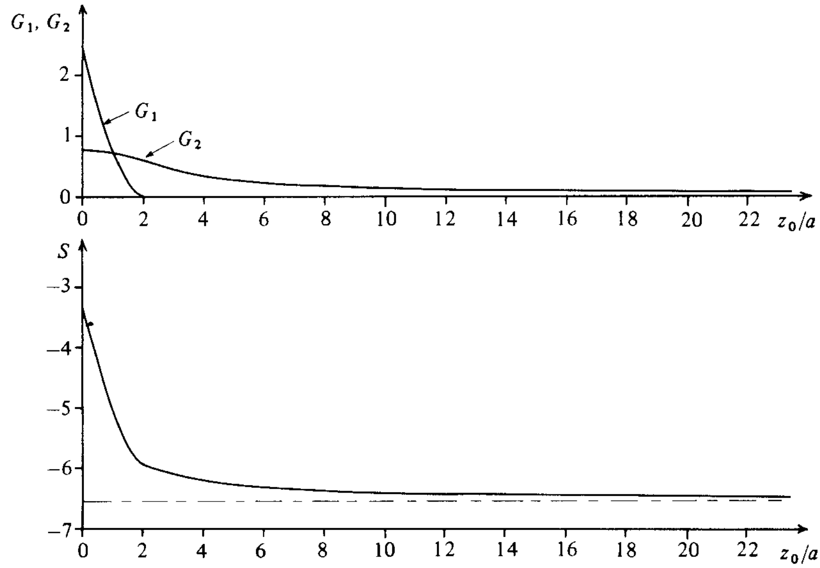

FIGURE 3. Case of a step profile in concentration at $z = 0$. The inhomogeneous sedimentation coefficients G_1 and G_2 and the coefficient $S = -6.55+G_1+G_2$ entering the average velocity of sedimentation $\bar{v}_p = v_{ps}\,(1+Sc)$ of a sphere centred at z_0.

coefficient S should be then half the one found by Batchelor (1972) for a homogeneous suspension. We check that

$$\left.\begin{aligned} 2G_1 &= 5, \\ 2G_2^{(0)} &= 1.546 \approx 1.55, \\ S &= -6.55+G_1+G_2 = -\frac{6.55}{2}. \end{aligned}\right\} \tag{6.30}$$

Actually, the result (6.28) for $G_2^{(0)}$ provides a more precise coefficient $S = -6.546$ the homogeneous suspension.

Another interpretation of the reduced backflow at the top of the cloud is the following: consider the flux of particles down across the plane $z = z_0$. It is equal to the velocity of particles multiplied by the total area S_p of the intersections of particles by the plane $z = z_0$. This area can be evaluated as

$$S_p = \mathscr{S}\int_{z=z_0-a}^{z_0+a} \sigma_p(z)\,P(z)\,\mathrm{d}z,$$

where $\sigma_p(z)$ is the area of the intersection of a sphere centred at z with the plane $z = z_0$, $P(z)$ is (as before) the probability density for this sphere to be centred at z, and $\mathscr{S}$ is the area of a horizontal cross-section of the container.

For a plane $z = z_0$ located in the homogeneous dispersion

$$S_p = n\mathscr{S}\int_{z=z_0-a}^{z_0+a} \sigma_p(z)\,\mathrm{d}z.$$

For the top plane $z_0 = 0$, the number of spheres contributing to S_p is divided by 2; thus S_p is divided by 2. The flux of particles downward, of the order of $S_p\,v_{ps}$, is half that of the homogeneous dispersion. The result for the backflow follows, since it is by definition equal to the flux of particles downward.

From the variation of the sedimentation coefficient S with z_0 (figure 3) it is seen that the influence of the inhomogeneity becomes small at about 10 radii under the top of the cloud. The influence of a sinusoidal inhomogeneity with a large amplitude (figure 1) looks comparatively larger, as for wavelengths of about 50 radii, S still changes by 10%. Thus the contributions from distant periodic inhomogeneities add up.

A consequence of the present model is that, for an inhomogeneous suspension, an initial step profile in concentration will begin to distort. As particles at the top of the cloud fall faster, they will subsequently catch up with the preceding particles and the concentration will begin to increase at the top. This behaviour is apparently not observed experimentally since a cloud of particles appears to fall steadily with a sharply defined upper limit and uniform concentration. However, the results of this section are based on assumption (6.1) which defines a given initial structure of doublets. Subsequently, doublets of closed spheres, which fall faster than single spheres, will not be replaced by doublets coming from above, in the case of the upper limit of the cloud. The concentration in closed doublets will decay there. This effect would induce a reduction in the average velocity of sedimentation, compensating the reduced backflow at the top of the cloud. More quantitative results concerning these effects should be obtained on the basis of the conservation equation for the pair probability distribution. We leave this question for future study.

7. Conclusions

A method has been presented which avoids the divergent integrals appearing in the averaging of hydrodynamic interactions in a dilute dispersion with vertical inhomogeneities. Its principle is to separate the physical effects in

(i) the direct influence of statistically independent spheres considered as point forces, for which the volume V containing the spheres is taken to be finite;

(ii) the statistical dependence and hydrodynamic interactions between spheres, for which the radius of a sphere is a significant lengthscale; the volume V containing the spheres there goes to infinity, but the integrals are convergent;

(iii) the volume effect of the independent spheres is an intermediate step for which both descriptions for finite and infinite V have to be used.

Steps (i) and (iii) are made possible by a consistent mathematical development on averaged point singularities in Stokes flow.

The average velocity of sedimentation of a sphere is obtained, (5.12), in terms of integrals involving the probability distribution for a single particle (i.e. the concentration) and the conditional probability distribution for pairs of particles. This result is valid under the assumptions that

(i) a quasi-ergodic hypothesis holds;

(ii) the probabilities can be expressed in terms of probability densities.

Necessary consequences of (ii) are that the particles should not adhere to anywhere with a non-zero probability, and that the quantity $P(\boldsymbol{x}_1|\boldsymbol{x}_0)-P(\boldsymbol{x}_1)$, representing the statistical dependence between both spheres of a doublet, should decay fast enough for (5.7) to hold.

The present paper is limited to vertical inhomogeneities, but the calculations might be extended to the more general case of a completely inhomogeneous suspension. Such a problem would involve more general equations for the problem of statistically independent Stokeslets plus Stokeslet quadrupoles in a container.

In this paper the probability distributions have not been calculated, and in §6 some forms of probability distributions have been assumed so as to evaluate the integrals in the result (5.12). The assumption (6.1) is used for the conditional probability density distribution. The initial behaviour of the suspension is considered, for given concentration profiles. It is found that, for a sinusoidal concentration profile, the effect of inhomogeneity upon the average velocity of sedimentation is comparatively larger than for a step function in concentration.

According to the model, both sinusoidal and step concentration profiles begin to distort. For the sinusoidal profile the rate of overturning is found to be an increasing function of the wavelength. In the case of the step profile the particles at the top of the cloud are found to fall faster than the ones in the rest of the cloud, this effect being limited to a layer about 10 radii below the top. To model the subsequent behaviour of the cloud, for future comparison with experiments, the assumption (6.1) should be replaced by a detailed study of an evolution equation for the pair distribution function.

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